Sequence of constrained annealed averages for one-dimensional disordered systems

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We introduce a systematic way of implementing a sequence of constrained annealed averages which converges to the quenched average in one-dimensional systems with dichotomic disorder. It is formulated in the context of products of random matrices where the constraints correspond to imposing that each type of n-ple of consecutive matrices appears with the correct frequency according to the given probability distribution in the annealed average. We apply the method to a one-dimensional disordered Ising model, where the constraints have the effect of preventing the disappearance of frustration.

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The idea of studying systems with quenched disorder by annealed models has a long history. In the framework of magnetic systems, Morita [1] noted that a quenched average is formally equivalent to an appropriate annealed average. Although this result is important from a theoretical point of view, it is not very useful for estimating a quenched average, since the equivalent annealed model has the same degree of difficulty as the original one.

A more practical approach is to consider an annealed version of the model where one physically relevant constraint is imposed: for instance, a concentration of impurities in the Ising model with diluted random couplings [2] frustration on elementary plaquettes in spin glasses [3,4], or the frequency of a matrix in the product of two independent random matrices [5]. In fact, by means of Lagrange multipliers, it is possible to introduce a method to perform annealed averages with an arbitrary number of constraints related to self-averaging intensive quantities [6]. Its thermodynamic interpretation allows one to distinguish between relevant and irrelevant constraints. Indeed the Lagrange multiplier can be regarded as the chemical potential related to the work necessary to freeze the constrained quantity. Irrelevant quantities are those which can be frozen without work, so that annealed and constrained annealed averages have the same value. Moreover this approach can be applied to a product of random transfer matrices and thus is useful not only for Ising models but also for quantum problems such as the localization in the Anderson model [7].

In this paper, we use the method of [6], in order to reproduce quenched averages by a convergent sequence of relevant constraints, in the case of dichotomic disorder in one dimension. In principle, the results might be extended to multinomial disorder distributions, and, more importantly, to higher dimensions via finite size scaling.

Let us briefly discuss the difference between quenched and annealed averages in the context of Ising models with partition function $Z_N(\beta)=\mathrm{Tr}e^{-\beta\mathcal{H}}$, where N is the number of spins, $\mathcal{H}$ is the Hamiltonian, and β is the inverse temperature. In the presence of disorder, Z_N itself is a random variable, although the typical free energy is given by the so-called quenched average (over the disorder)

$$f_N=-\frac{1}{\beta N}\overline{\ln Z_N}\ . \tag{1}$$

In the thermodynamic limit $N\to\infty$, all disorder realizations (with the exception of a set of zero probability measure) have the same free energy

$$\lim_{N\to\infty}-\frac{1}{\beta N}\ln Z_N=\lim_{N\to\infty}f_N=f\ . \tag{2}$$

This property is called the self-average, since $(\ln Z_N)/N$ becomes a nonrandom quantity for $N\to\infty$. The calculation of the quenched free energy is a difficult problem even in simple one-dimensional models. In practice, it is much easier to estimate the quenched average $\overline{\ln Z}$ by the annealed average $\ln\overline{Z}$, obtaining a lower bound of f. However, annealed averages are often very bad approximations and, more importantly, can fail to describe even the qualitative aspects of a disordered system. This is due to the fact that in an annealed average the disorder variables (for instance couplings between spins or magnetic fields in a random Ising model) can arrange themselves to minimize the free energy, while in a quenched model they are frozen in some given realizations. The latter is a realistic assumption, since, in general, the disorder variables have much longer evolution times than the thermodynamic variables (e.g., spins). As a consequence, a disorder realization of a system can be frustrated. In other terms, there exist realizations where different spin configurations are energetically equivalent at sufficiently low temperatures, and the system has a problematic choice.

An analogous situation characterizes a well-known mathematical problem: the product of random matrices [7]. Let us suppose a matrix $\mathbf{A}_\eta$ that depends on a random variable η, and consider a product of N of these matrices, $\mathbf{A}_{q_1},\ldots,\mathbf{A}_{\eta N}$, where $\eta_1,\ldots,\eta_N$ are independent identically distributed random variables. The so-called maximum Lyapunov exponent

$$\lambda=\lim_{N\to\infty}\frac{1}{N}\overline{\ln\left[\mathrm{Tr}\prod_{i=1}^{N}\mathbf{A}_{\eta_i}\right]} \tag{3}$$

represents a characteristic quantity of the random product (the overbar denotes the average over the $\{\eta_i\}$ distribution). The multiplicative ergodic theorem of Oseledec [8] assures us that the following limit exists and is unique for almost all $\{\eta_i\}$ realizations (apart from a set of zero probability measure)

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \ln \left[\mathrm{Tr} \prod_{i=1}^{N} \mathbf{A}_{\eta_i} \right] \quad \text{for almost all } \{\eta_i\} \ . \tag{4}$$

Comparing (1) and (3), one realizes the common points between disorder systems and products of random matrices. In fact, the maximum Lyapunov exponent λ, as well as the free energy f, is a difficult quantity to compute. Following the comparison, one can introduce the annealed version L_0 of the Lyapunov exponent:

$$L_0 = \lim_{N\to\infty} \frac{1}{N} \ln \left[\overline{\mathrm{Tr} \prod_{i=1}^{N} \mathbf{A}_{\eta_i}} \right] , \tag{5}$$

which is an upper bound of λ, but it is often inadequate even as a qualitative approximation.

As a specific example of a product of random matrices, we study the one-dimensional Ising model with Hamiltonian

$$\mathcal{H} = -\sum_i J\sigma_i\sigma_{i+1} - \sum_i h_i\sigma_i \ , \tag{6}$$

where $J=1$ is the positive coupling, $h_i = H + h\eta_i$ is a random field ($H>0$, $h>0$), and $\{\eta_i\}$ are independent random variables which assume the values $\eta_i = \pm 1$ with equal probability. Let us stress that in this paper we do not want to discuss the problem of random field Ising models and phase transitions, which is widely discussed in the literature.

All the following arguments can be repeated in a straightforward way for any product of binomial independent random matrices. Indeed, we can write the quenched free energy of the model as the product of transfer random matrices:

$$f = -\lim_{N\to\infty} \frac{1}{\beta N} \ln \left[\overline{\mathrm{Tr} \prod_{i=1}^{N} \mathbf{A}_{\eta_i}} \right] , \tag{7}$$

where

$$\mathbf{A}_{\eta_i} = \begin{bmatrix} e^{\beta(J+H+h\eta_i)} & e^{\beta(-J+H+h\eta_i)} \\ e^{\beta(-J-H-h\eta_i)} & e^{\beta(J-H-h\eta_i)} \end{bmatrix} .$$

The quenched free energy is thus given by $f = -\lambda/\beta$, where λ is the maximum Lyapunov exponent of the matrix product.

The method of constrained annealed averages allows one to obtain an upper bound L of the Lyapunov exponent:

$$L = \min_{\mu} \lim_{N\to\infty} \frac{1}{N} \ln \left[\overline{e^{-N\mu\cdot(\alpha-\bar{\alpha})} \mathrm{Tr} \prod_{i=1}^{N} \mathbf{A}_{\eta_i}} \right] \geq \lambda \ , \tag{8}$$

where $\mu = \{\mu_1, \ldots, \mu_M\}$ are M Lagrange multipliers which fix M self-averaging intensive quantities $\alpha = \{\alpha_1, \ldots, \alpha_M\}$ to their mean values $\bar{\alpha} = \{\overline{\alpha_1}, \ldots, \overline{\alpha_M}\}$ in the limit $N\to\infty$, so that the set of disorder realizations selected by the constraints is an ensemble of full η probability measures.

In our case, the simplest quantity to be considered is

$$\alpha_1 = \frac{1}{N} \sum_{i=1}^{N} \eta_i \ , \tag{9}$$

with $\overline{\alpha_1} = 0$. This constraint is derived by the large number law and leads to the estimate

$$L_1 = \min_{\mu_1} \ln[\text{max. eigenvalue } \mathbf{G}_1(\mu_1)] \ , \tag{10}$$

with

$$\mathbf{G}_1(\mu_1) = \tfrac{1}{2}(\mathbf{A}_1 e^{-\mu_1} + \mathbf{A}_{-1} e^{\mu_1}) \ .$$

The previous expression of $\mathbf{G}_1(\mu_1)$ is obtained by a calculation which is identical to an annealed average without constraints; in other terms, the argument of the logarithm in (8) is equal to the Nth power of a transfer matrix $\mathbf{G}_1(\mu_1)$ corresponding to a particular one-dimensional nonrandom system, depending on the multiplier μ_1. Its eigenvalue is given by the secular equation, which requires the solution of a second order polynomial, while the minimization over μ_1 corresponds to the solution of a fourth order polynomial.

In the case of an average magnetic field $H=0$, it is possible to show that the value of the multiplier that realizes the minimum is $\mu_1^* = 0$. This means that imposing $\alpha_1 = 0$ does not require any thermodynamic work, and there is no difference between annealed averages with or without constraint, as a consequence of the invariance of the Hamiltonian (6) under a simultaneous inversion of all fields and spins at $H=0$.

In general, the annealed average without constraints,

$$L_0 = \ln[\text{max. eigenvalue } \mathbf{G}_1(\mu_1 = 0)] \ ,$$

can be very different from λ, while L_1 is often a reasonable estimate, as shown in Fig. 1. In Fig. 2, we show the relative error of L_1 with respect to the numerical value of λ obtained by a numerical simulation, at varying H and h with inverse temperature $\beta=1$. The error is maximum when the parameters satisfy the relation $h = H + 2J$.

In the particular context of our model, we can give a simple justification of this result. The condition $h = H + 2J$ is one which frustrates a spin in a site. Indeed, consider the case of a site i with random field $h_i = H - h$ (that is, $\eta_i = -1$) and with two spins up in the two neighbors sites; see Fig. 3(a). The energy of the configuration $\sigma_i + 1$ is $E_+ = -2J - H + h$, while if $\sigma_i = -1$ the energy is $E_- = -E_+$. The two configurations are degenerate when $E_+ = E_- = 0$. However, the constrained annealed system can still move its random fields to minimize free energy, though the number of sites where $\eta_i = +1$ is fixed to be $N/2$. It follows that this does not exhibit the local frustration of the quenched system, since it can separate itself into two parts (e.g., the first $N/2$ sites with $\eta_i = +1$, followed by a sequence of $N/2$ variables $\eta_i = -1$). In this situation, if the temperature is not too high, the constraint

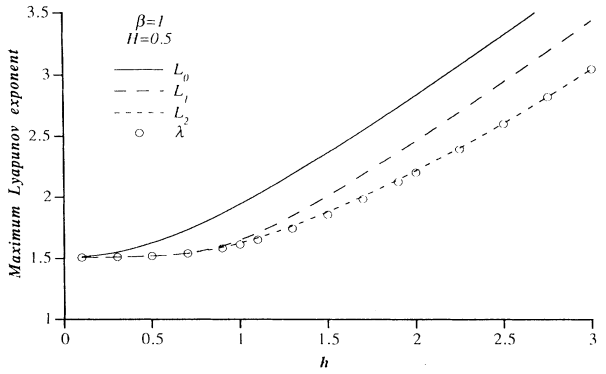

FIG. 1. Annealed averages without constraints L_0 (full line), and with constraints L_1 (dashed line) and L_2 (dotted line) compared with the numerical results for the maximum Lyapunov exponent (circles) for $\beta=1$ and $H=0.5$ as functions of h. The size of the circles does not represent the error of the numerical estimates, which is much lower.

$\alpha_1 \equiv (1/N)\sum \eta_i = 0$ is not sufficient to obtain a good qualitative description of the quenched system. In fact, we must impose a further constraint on the system that does not permit the formations of these islands of random variables of the same sign. Such islands lead to a predominance of $\eta_i\eta_{i+1}=+1$ with respect to $\eta_i\eta_{i+1}=-1$ while, in a typical realization of the quenched system, the law of large numbers implies that

$$\alpha_2 \equiv \frac{1}{N}\sum_i \eta_i\eta_{i+1}=0 \ . \tag{11}$$

In the more general context of a product of random matrices, and without any reference to frustration, imposing the constraints $\alpha_1=0$ and $\alpha_2=0$ is equivalent to considering only sequences where the number of every possible pair of consecutive matrices (i.e., $\mathbf{A}_{+1}\mathbf{A}_{+1}, \mathbf{A}_{+1}\mathbf{A}_{-1}, \ldots$) appears with a frequency which is equal to the probability in the thermodynamic limit. In our case each type of couple has probability $\frac{1}{4}$.

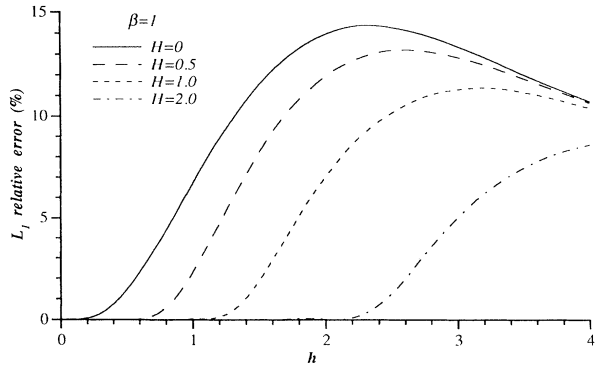

FIG. 2. Ising model (6) with $\beta=1$: relative difference between L_1 and the numerical value of the Lyapunov exponent λ, as functions of H and h.

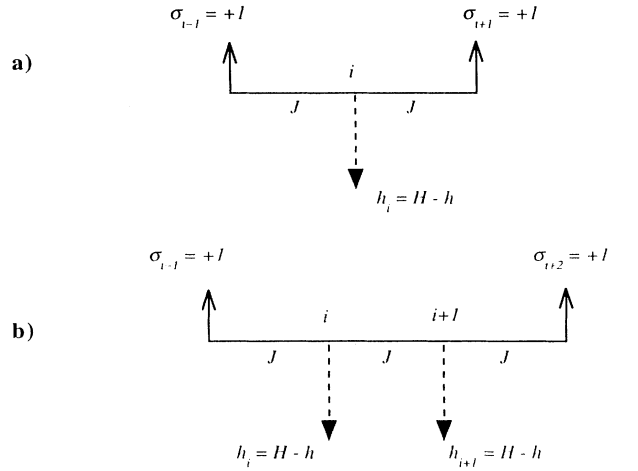

FIG. 3. Illustration of a frustrated spin configuration in (a) in elementary segment of one site; and (b) a segment of two sites. The spins are represented by the arrows; the dashed vertical line represents the random field $H-h$; and the solid line represents the positive J coupling.

The estimate of the Lyapunov exponent with two constraints is

$$L_2 = \min_{\mu_1,\mu_2} \lim_{N\to\infty} \frac{1}{N} \ln \left[\overline{e^{-N(\mu_1\alpha_1+\mu_2\alpha_2)} \mathrm{Tr} \prod_{i=1}^{N} \mathbf{A}_{\eta_i}} \right] \leq L_1 \ . \tag{12}$$

As in the case of one constraint [see formula (10)], L_2 is given by the minimization over the Lagrange multipliers μ_1 and μ_2 of the logarithm of the maximum eigenvalue of the 4×4 transfer matrix

$$L_2 = \min_{\mu_1,\mu_2} \ln[\text{max. eigenvalue } \mathbf{G}_2(\mu_1,\mu_2)] \ , \tag{13}$$

with

$$\mathbf{G}_2(\mu_1,\mu_2) = \frac{1}{2} \begin{bmatrix} \mathbf{A}_{+1} e^{-\mu_1-\mu_2} & \mathbf{A}_{+1} e^{-\mu_1+\mu_2} \\ \mathbf{A}_{-1} e^{+\mu_1+\mu_2} & \mathbf{A}_{-1} e^{+\mu_1-\mu_2} \end{bmatrix} .$$

In Fig. 1 one sees that the new constraint $\alpha_2=0$ permits us to obtain a very accurate estimate of the Lyapunov exponent. Figure 4 gives the relative error of L_2 with respect to the Lyapunov exponent as a function of H and h ($\beta=1$): it is always below 1.5%. With two constraints, the error is at a maximum when the parameters verify the relation $h=H+J$.

We can again understand this finding by relating it to a frustrated situation, now involving the spins on two sites [see Fig. 3(b)]. Indeed, consider two random fields $h_i=h_{i+1}=H-h$ and two spins up on the neighbor sites. If $h=H+J$, there exist two energetically equivalent configurations where the two spins on the sites (i and $i+1$) have the same direction (up or down). The annealed system with constraints $\alpha_1=0$ and $\alpha_2=0$ can avoid this frustrated situation, selecting configurations of islands of size $N/4$: two of them with fields equal to $H+h$ and

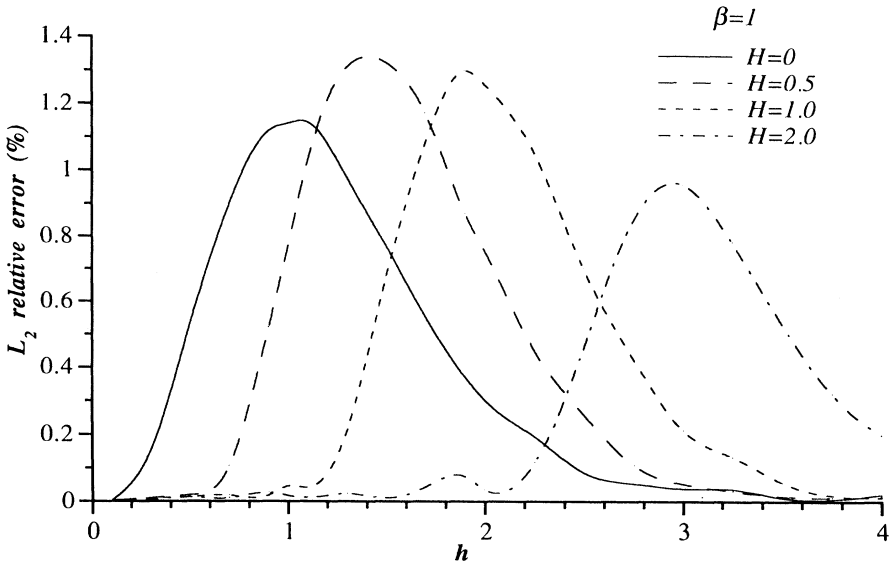

FIG. 4. Ising model (6) with $\beta=1$: relative difference between L_2 and numerical value of the Lyapunov exponent λ, as functions of H and h.

$H-h$, respectively, and the other two islands with variables $\{\eta_i\}$ which have alternated signs.

Our constraints can easily be generalized to prevent the corresponding annealed systems from choosing particular disorder realizations of equilibrium which ignore frustration over segments with larger and larger number of sites. For a segment of size $n-1$, this goal can be achieved by imposing that in the product of random matrices (7), each type of n-ple of consecutive matrices appears with the right frequency (in the thermodynamic sense), e.g., among the triples $\mathbf{A}_{+1}\mathbf{A}_{+1}\mathbf{A}_{+1}$ has to appear $N/8$ times when $N\to\infty$.

In terms of the η's, we have to perform annealed averages with constraints on all the 2^{n-1} quantities of the type

$$\frac{1}{N}\sum_{i=1}^{N}\eta_i\eta_{i+s_1}\cdots\eta_{i+s_k}\ , \tag{14}$$

where $\{s_1,\ldots,s_k\}$ are k integers such that: $1\le s_1<s_2<\cdots<s_k\le n-1$ [9]. The choice $k=0$ corresponds to α_1 (9). For example, for $n=3$ we have to introduce two new Lagrange multipliers μ_3 and μ_3 related to

$$\begin{aligned}\alpha_3&\equiv\frac{1}{N}\sum_{i=1}^{N}\eta_i\eta_{i+2}\ ,\\ \alpha_4&\equiv\frac{1}{N}\sum_{i=1}^{N}\eta_i\eta_{i+1}\eta_{i+2}\ ,\end{aligned} \tag{15}$$

together with μ_1 and μ_2 related to α_1 and α_2, respectively, given by (9) and (11).

In practice, the problem can be reduced to the determination of the largest eigenvalue of a matrix $\mathbf{G}_n(\mu)$ of size $2^n\times 2^n$, as provided in the Appendix. The upper bound L_n is then given by a minimization on the vector of multipliers $\mu=\{\mu_1,\mu_2,\ldots\}$:

$$L_n=\min_{\mu}\ln[\text{max. eigenvalue }\mathbf{G}_n(\mu)]\ . \tag{16}$$

A moment of reflection shows that

$$\lambda\le L_n\le L_{\bar n}\quad\text{for } n>\bar n\ . \tag{17}$$

The computation of L_3 for $H=0.5$ and $\beta=1$ is shown in Fig. 5, where one sees that there is a considerable improvement over the estimate L_2. However, the frustration on segments larger than $n=2$ still produces a difference between L_3 and λ which attains its maximum (0.2%) when $h\sim 1$.

For a vanishing magnetic field $H=0$, as a generalization of what discussed for $n=1$, it is possible to prove by symmetry arguments that the constraints on quantities of type (14) with products of an odd number of η's are irrelevant, so that the corresponding Lagrange multipliers are zero [9].

More important, the constraints on the n-ples of matrices that we have considered are very general in a wide class of functions. To be explicit, let us consider the case of the couples. One can decide to constrain two other quantities to their mean value, let us say the quantities γ_1 and γ_2 instead of α_1 and α_2 given by (9) and (11). Indeed, if we consider intensive quantities of the kind

$$\begin{aligned}\gamma_1&=\frac{1}{N}\sum_{i=1}^{N}\phi_1(\eta_i,\eta_{i+1})\ ,\\ \gamma_1&=\frac{1}{N}\sum_{i=1}^{N}\phi_2(\eta_i,\eta_{i+1})\ ,\end{aligned} \tag{18}$$

where $\phi_1(\eta_i,\eta_{i+1})$ and $\phi_2(\eta_i,\eta_{i+1})$ are two independent sufficiently regular functions of two variables, then constraining the quantities (18) is equivalent to imposing $\alpha_1=0$ and $\alpha_2=0$ by virtue of the dichotomic properties of the η's. In other terms, the two ensembles of full η probability measure selected by the different constraints $\{\alpha_1=0,\alpha_2=0\}$ or $\{\gamma_1=\overline{\gamma_1},\gamma_2=\overline{\gamma_2}\}$ coincide. The generalization to n-ples with $n>2$ is immediate. The sequence of estimates $L_0,L_1,\ldots,L_n$ converges to λ, since the Lyapunov exponent of a specific realization of the product of random matrices can be written as a function

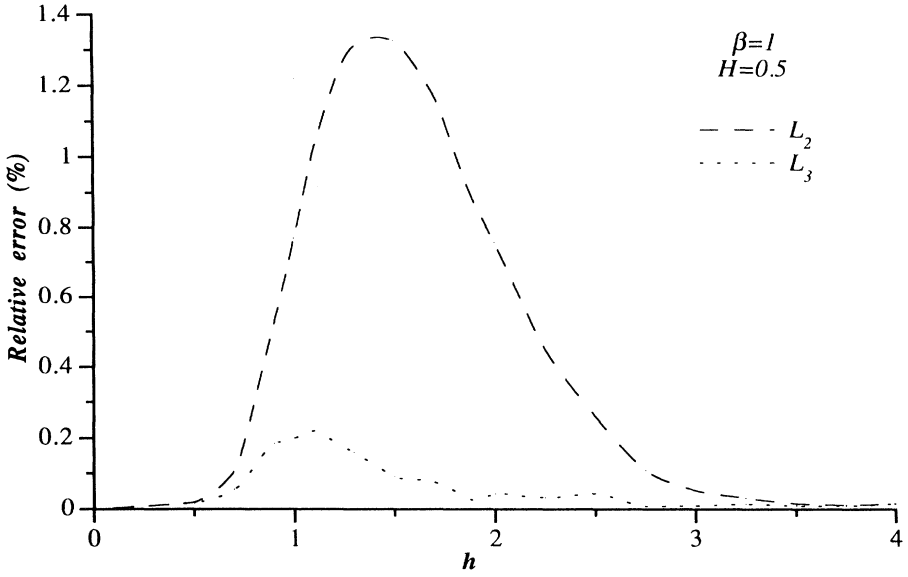

FIG. 5. Ising model (6) with $H=0.5$ and $\beta=1$: relative difference between L_3 and numerical value of the Lyapunov exponent λ (dotted line). For comparison, the dashed line indicates the relative difference between L_2 and λ. The relative error is maximum (0.2%) at $h\sim 1$.

only of the variables of type (14), such as $\alpha_1,\alpha_2,\ldots$, and so on. Note that in general the number of these variables is infinite. A rigorous proof of this statement can be found in the Appendix of [10].

In the Ising model, one sees that when $n\to\infty$ the quenched system has the same degree of frustration as the constrained annealed system, since the former as well as the latter can be regarded as the superposition of frustrated segments of different sizes $\tilde{n}\le n$. The convergence of L_n toward the Lyapunov exponent seems to be extremely fast, probably exponent in n, at least for nonzero temperature. For instance, in the random field Ising model at inverse temperature $\beta=1$, the maximum relative error is 30% for L_0, 10% for L_1, 1.5% for L_2, and 0.2% for L_3.

The method of constrained annealed averages is also able to give good estimates of other thermodynamical quantities, such as the quenched magnetization, defined as

$$m=\lim_{N\to\infty}\overline{\left\langle\frac{1}{N}\sum_{i=1}^{N}\sigma_i\right\rangle}\ , \tag{19}$$

where $\langle\cdots\rangle$ represents the thermodynamic average. The magnetization m is related to the derivative of the free energy

$$m=-\frac{\partial f}{\partial H}=\frac{1}{\beta}\frac{\partial\lambda}{\partial H}\ , \tag{20}$$

so that the difficulty of calculating the magnetization is almost the same as for f (or λ). It is thus natural to introduce an estimate m_n of the quenched magnetization, substituting λ for an upper bound L_n in (20):

$$m_n=\frac{1}{\beta}\frac{\partial L_n}{\partial H}\ . \tag{21}$$

However, m_n is not an upper bound of m different from L_n and λ, although we expect that m_n is quickly convergent to m at increasing n.

To be explicit, let us consider

$$m_2=\frac{1}{\beta}\frac{\partial}{\partial H}\lim_{N\to\infty}\frac{1}{N}\times\ln\overline{\left[\mathrm{Tr}\left[\prod_{i=1}^{N}\mathbf{A}_{\eta_i}\right]e^{-N(\mu_1^*\alpha_1+\mu_2^*\alpha_2)}\right]}\ , \tag{22}$$

where (μ_1^*,μ_2^*) are the values of the Lagrange multipliers that realize the minimum (12). After simple algebraic manipulations, one obtains

$$m_2=\lim_{N\to\infty}\frac{\overline{\mathrm{Tr}\left[\sigma_z\prod_{i=1}^{N}\mathbf{A}_{\eta_i}\right]e^{-N(\mu_1^*\alpha_1+\mu_2^*\alpha_2)}}}{\overline{\mathrm{Tr}\left[\prod_{i=1}^{N}\mathbf{A}_{\eta_i}\right]e^{-N(\mu_1^*\alpha_1+\mu_2^*\alpha_2)}}}\ , \tag{23}$$

where

$$\sigma_z=\begin{bmatrix}+1 & 0\\ 0 & -1\end{bmatrix}$$

is a Pauli matrix. As previously discussed, the denominator in (23) can be reduced to the trace of the Nth power of the transfer matrix $\mathbf{G}_2(\mu_1^*,\mu_2^*)$. It is easy to show that even the numerator can be expressed in a similar way, so that

$$m_2=\lim_{N\to\infty}\frac{\mathrm{Tr}(\mathbf{D}_2\mathbf{G}_2^N(\mu_1^*,\mu_2^*))}{\mathrm{Tr}(\mathbf{G}_2^N(\mu_1^*,\mu_2^*))}$$

$$=(\mathbf{S}_2^{-1}(\mu_1^*,\mu_2^*)\mathbf{D}_2\mathbf{S}_2(\mu_1^*,\mu_2^*))_{1,1}\ , \tag{24}$$

where $\mathbf{D}_2$ is the 4×4 matrix

$$\mathbf{D}_2=\begin{bmatrix}\sigma_z & \mathbf{0}\\ \mathbf{0} & \sigma_z\end{bmatrix}\ ,$$

and $\mathbf{S}_2(\mu_1^*,\mu_2^*)$ is the matrix of the similarity transformation which diagonalizes $\mathbf{G}_2(\mu_1^*,\mu_2^*)$, such that

$$(\mathbf{S}_2^{-1}(\mu_1^*,\mu_2^*)\mathbf{G}_2(\mu_1^*,\mu_2^*)\mathbf{S}_2(\mu_1^*,\mu_2^*))_{1,1}=\exp L_2 \ .$$

From the computational point of view, m_2 can easily be evaluated, once L_2 is known.

In the general case, for m_n one obtains an expression similar to (24), where $\mathbf{S}_n(\mu)$ is the $2^n\times 2^n$ matrix of the similarity transformation which diagonalizes the related transfer matrix $\mathbf{G}_n(\mu)$, and $\mathbf{D}_n$ is a diagonal matrix with entries equal to ± 1, alternated along the diagonal. In Fig. 6, we compare $m_0,\ldots,m_3$ with the values of quenched magnetization obtained by derivating a polynomial fit of numerical calculation of the free energy with respect to h, for $h=2$ and $\beta=1$, as functions of H.

In conclusion, we have introduced a general tool to estimate the Lyapunov exponent λ as a product of binomial random matrices by a nonincreasing sequence of bounds L_n. These bounds are obtained by annealed averages with constraints related to the frequency of all the $\tilde{n}$-ple of consecutive matrices, with $\tilde{n}\le n$.

In particular we have applied the procedure to a one-dimensional Ising model with binomial disorder, where we have obtained extremely accurate estimates of the quenched free energy and the magnetization, more precise than the numerical results. In this model the effect of the constraints has a simple physical interpretation: it prevents the disappearance of frustration on larger and larger segments of contiguous sites. These segments are the one-dimensional analog of plaquettes of larger and larger perimeters in spin glasses with dimension $d\ge 2$, where such a procedure was suggested by Toulouse and Vannimenus [3]. However, in that case, an analytic approach is very difficult (see, e.g., [10,11] for some partial results) and only Monte Carlo calculations with the constraints related to the elementary plaquette (the plaquette of smallest size) have been performed [4], as far as we know. On the other hand, our type of constraints might be useful in some disordered models in dimension $d=2$ or 3, by considering strips or squares on the lattice, and the related transfer matrices. Some work is in progress along this line.

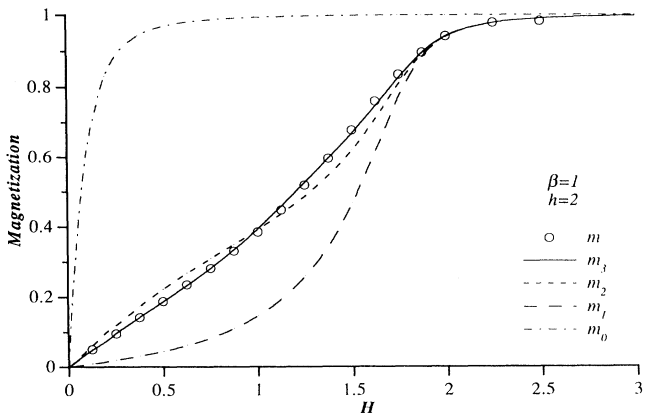

FIG. 6. Ising model (6) with $\beta=1$: numerical results of the quenched magnetization m (circles) as functions of H, compared with the annealed estimates m_0, m_1, m_2, and m_3, at $h=2$. The size of the circles does not represent the error of the numerical estimates.

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APPENDIX

In this appendix we prove that the argument of the logarithm in (8) is equivalent to the trace of the Nth power of an appropriate transfer matrix $\mathbf{G}(\mu)$, i.e.,

$$\overline{e^{-N\mu(\alpha-\bar{\alpha})}\mathrm{Tr}\prod_{i=1}^{N}\mathbf{A}_{\eta_i}}=\mathrm{Tr}\mathbf{G}^N(\mu) \tag{A1}$$

if the quantities $\alpha=\{\alpha_1,\ldots\}$ are of type (14). The generalization of the following results to the case of multinominal variables $\{\eta_i\}$ related to matrices $\mathbf{A}_{\eta_i}$ of a size greater than 2×2 is straightforward, and can be found in [9].

As previously discussed, to impose that the frequency of every different n-ple of consecutive matrices is fixed by the law of large numbers, one has to take into account every possible quantity (14) with $k\le n-1$. With this requirement, it is easy to show that the left-hand side of (A1) can be written in the form:

$$\overline{e^{-N\mu(\alpha-\bar{\alpha})}\mathrm{Tr}\prod_{i=1}^{N}\mathbf{A}_{\eta_i}}$$

$$=\mathrm{Tr}\sum_{\eta_1=\pm 1,\ldots,\eta_N=\pm 1}\prod_{i=1}^{N}\mathbf{g}(\eta_i,\ldots,\eta_{i+n-1}) \ , \tag{A2}$$

where $\mathbf{g}(\eta_i,\ldots,\eta_{i+n-1})$ is a 2×2 matrix depending on n variables $\{\eta_i,\ldots,\eta_{i+n-1}\}$. For instance, for $n=1$ (the case of one constraint $\alpha_1=0$) one has

$$\mathbf{g}_1(\eta_1)=\tfrac{1}{2}\mathbf{A}_{\eta_i}e^{-\mu_1\eta_1} \ ,$$

and result (A1) is trivial with

$$\mathbf{G}_1(\mu_1)=\sum_{\eta_1=\pm 1}\mathbf{g}_1(\eta_1) \ .$$

$\mathbf{G}_1(\mu_1)$ is thus a matrix of the same size of $\mathbf{g}_1(\eta_1)$.

The key point is that when $\mathbf{g}$ depends on more than one variable η_i, we can write a transfer matrix $\mathbf{G}$ (we now omit the dependence on μ) of size larger than $\mathbf{g}(\eta_i,\ldots,\eta_{i+n-1})$, although many entries of $\mathbf{G}$ are zero. The matrix $\mathbf{G}$ should be built up as a block matrix, starting from the different matrices $\mathbf{g}(\eta_i,\ldots,\eta_{i+n-1})$. From the parameters $\{\eta_i,\ldots,\eta_{i+n-1}\}$ we obtain the arrow and column index of the single block, in such a way that the former becomes the latter when $i\to i+1$. We thus obtain a 2×2 block of $\mathbf{G}$ as

$$\mathbf{G}_{(\eta_i,\ldots,\eta_{i+n-2}),(\eta_{i+1},\ldots,\eta_{i+n-1})}=\mathbf{g}(\eta_i,\ldots,\eta_{i+n-1}) \ .$$

The remaining blocks are 2×2 zero matrices, so that one has

$$\mathbf{G}_{(\eta_i,\ldots,\eta_{i+n-2}),(e_i,\ldots,e_{i+n-2})}=\mathbf{g}(\eta_i,\ldots,\eta_{i+n-2},e_{i+n-2}) \times \prod_{s=i+1}^{i+n-2}\delta_{\eta_s,e_{s-1}}, \quad \text{(A3)}$$

where $\delta_{a,b}$ is the 2×2 identity matrix if $a=b$, and the zero matrix otherwise. In conclusion, $\mathbf{G}$ is a matrix of size $2^n\times 2^n$, and each arrow has only four non-zero entries (only for $n=2$ are all the elements of $\mathbf{G}$ nonzero).

Using (A3), the trace of $\mathbf{G}^N$ is given by

$$\mathrm{Tr}\sum_{\xi_1^{(1)},\ldots,\xi_{n-1}^{(1)}}\cdots\sum_{\xi_1^{(N)},\ldots,\xi_{n-1}^{(N)}}\left[\prod_{i=1}^{N}\prod_{s=2}^{n-1}\delta_{\xi_s^{(i)},\xi_{s-1}^{(i+1)}}\right]\left[\prod_{i=1}^{N}\mathbf{g}(\xi_1^{(i)},\ldots,\xi_{n-1}^{(i)},\xi_{n-1}^{(i+1)})\right]. \quad \text{(A4)}$$

It is easy to see that in (A4) there are N groups of $n-1$ indices ξ which are equal, because of the various δ matrices. In particular, one has

$$\xi_s^{(i)}=\xi_1^{(i+s-1)}\begin{cases}\forall s=1,\ldots,n-1\\ \forall i=1,\ldots,N\end{cases}$$

(periodic boundary conditions are assumed), and, as a consequence,

$$\mathrm{Tr}\mathbf{G}^N=\mathrm{Tr}\sum_{\xi_1^{(1)},\ldots,\xi_1^{(N)}}\prod_{i=1}^{N}\mathbf{g}(\xi_1^{(i)},\ldots,\xi_1^{(i+n-1)}). \quad \text{(A5)}$$

Recalling (A2), this concludes the proof.

Let us finally write the form of $\mathbf{G}$ for $n=2$ (the constraint on the couples of matrices) and 3 (triples) which are used in this paper to determine L_2 and L_3. In the former case, we have the 4×4 matrix

$$\mathbf{G}_2(\mu_1,\mu_2)=\begin{bmatrix}\mathbf{g}_2(+1,+1) & \mathbf{g}_2(+1,-1)\\ \mathbf{g}_2(-1,+1) & \mathbf{g}_2(-1,-1)\end{bmatrix},$$

with the 2×2 block matrices

$$\mathbf{g}_2(\eta_1,\eta_2)=\tfrac{1}{2}\mathbf{A}_{\eta_1}e^{-\mu_1\eta_1-\mu_2\eta_1\eta_2},$$

while in the latter case we have the 8×8 matrix

$$\mathbf{G}_3(\mu_1,\mu_2,\mu_3,\mu_4)=\begin{bmatrix}\mathbf{g}_3(+1,+1,+1) & \mathbf{g}_3(+1,+1,-1) & 0 & 0\\ 0 & 0 & \mathbf{g}_3(+1,-1,+1) & \mathbf{g}_3(+1,-1,-1)\\ \mathbf{g}_3(-1,+1,+1) & \mathbf{g}_3(-1,+1,-1) & 0 & 0\\ 0 & 0 & \mathbf{g}_3(-1,-1,+1) & \mathbf{g}_3(-1,-1,-1)\end{bmatrix},$$

with the 2×2 block matrices

$$\mathbf{g}_3(\eta_1,\eta_2,\eta_3)=\tfrac{1}{2}\mathbf{A}_{\eta_1}e^{-\mu_1\eta_1-\mu_2\eta_1\eta_2-\mu_3\eta_1\eta_3-\mu_4\eta_1\eta_2\eta_3}.$$

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